

A Stability Index for Local Effectivity Functions

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Abstract We study the structure of unstable local effectivity functions defined for n players and p alternatives. A stability index based on the notion of cycle is introduced. In the particular case of simple games, the stability index is closely related to the Nakamura Number. In general it may be any integer between 2 and p . We prove that the stability index for maximal effectivity functions and for maximal local effectivity functions is either 2 or 3.

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1 Introduction

Stability is a highly desirable property for political systems. The modeling of political interactions has to take stability requirements into account. In coalitional models, stability is defined as the possibility of achieving, for any preference profile, a state that no coalition would oppose. In strategic models, this amounts to the existence, for any preference profile, of an equilibrium (solvability). However it is commonly known that most political systems are unstable in this sense. In mathematical social sciences, results known as impossibility theorems reflect the fact that stability (or solvability) is rather hard to obtain. Therefore it is interesting to investigate the properties of *unstable* mechanisms.

As a first step in this investigation we introduce a stability index. This is an integer that measures the likelihood of the emergence of a situation where the power mechanism is unable to reach a stable outcome. Such an index may be used to compare political institutions or collective choice mechanisms. Power distributions with a lower index will be deemed less stable than those with a higher index. Other indices of stability exist in the literature, but they are concerned with games not game forms, and they are defined for some equilibrium point. Our index, on the other hand, is defined for power mechanisms and some solution concept, not for concrete situations generated by some preference profile. It is determined by the configurations that may produce instability.

Although this investigation can be carried out in a far more general setting (see Abdou and Keiding (2003) for the general notion of effectivity structure), we shall focus in this study on the so-called local effectivity functions. While effectivity functions appear naturally in the study of implementation theory (Moulin and Peleg, 1982; Peleg and Winter, 2002; Peleg, 2004; Peleg and Peters, 2008), as well as in the theory of Constitutions and Rights (Peleg, 1998), local effectivity functions are closely related to the solvability problem (Abdou, 1995, 2000). In their investigation of game form solvability (e.g. for Nash or strong Nash equilibrium), Abdou and Keiding (2003) pointed out two new aspects related to the power distribution: (1) the dependence of this power on the current state (the local aspect), (2) the interactive character of the power (involving the simultaneous action of many coalitions). In this paper we limit ourselves to the local aspect only, but instead of deriving our object from a strategic game form, we define it abstractly. Like an effectivity function, a local effectivity function describes the power of coalitions to achieve an outcome in some subsets of alternatives, but unlike an effectivity function, this power may depend on the current state. Our choice to restrict our attention on this object is justified by its simplicity, its natural interpretation and the elegance of the properties involved in its analysis. However, despite this restriction,

the local aspect adds an original ingredient and reveals new properties that cannot be articulated if we limit the study to effectivity functions. Stability of local effectivity functions is defined as the non-vacuity of the core for all preference profiles.

Necessary and sufficient conditions for stability of local effectivity functions may be deduced as a particular case of stability of effectivity structures (Abdou and Keiding, 2003, Theorem 6) and therefore characterization of stability is not the central question of this article. Here our main objective is to further investigate, *what can be said when the local effectivity function is unstable*. The notion of cycle lies at the heart of stability characterization. This fact, known since Condorcet, turns out to be very general. A cycle (definition 3.1) is a sequence of potential objections that obeys a combinatorial property that guarantees their compatibility. The existence of a cycle is equivalent to the existence of some profile for which any potential state is opposed by some coalition. Therefore, to study instability we must explore the structure of cycles. Defining a stability index is a first step in that direction.

The idea that lies behind our notion of stability index is that small cyclic configurations are more likely to emerge than larger ones. Any such configuration includes potentially (1) the formation of a coalition structure and (2) the elaboration of a coordinated action among the members of each coalition in order to oppose some package of alternatives. Since the formation of coalitions and the coordination of actions are costly, we implicitly postulate that configurations with a small number of coalitions and concerted actions are more likely to surface than larger ones. In accordance with this idea, our index will be defined precisely as the minimal length of all possible cycles ($+\infty$ in case of stability). This number does not exhaust all the features of instability, but does provide a meaningful classification of instability types. If the cardinality of the alternative set is p , the stability index can be any integer between 2 and p . If the power distribution is such that the index is 2, then instability takes a particularly simple form: alternatives can be partitioned into two aggregates, or two major issues, on which the society is split, and the power of coalitions allowed by the rules is such that both issues can be opposed and neither one can be forced (political stalemate or deadlock). Situations with a low index will be illustrated by examples from politics.

As direct applications of this definition, we identify the index of some subclasses of local effectivity functions. In the case of simple games, our index can be viewed as the analog of the Nakamura number (Nakamura, 1979). Indeed, when the stability index is finite then it coincides with the Nakamura number (Corollary 4.8). However, it should be emphasized that we provide a new interpretation of this notion. Classically the Nakamura

number is used as a criterion for stability : A simple game acting on some alternative set is stable if and only if the number of alternatives is strictly lower than the Nakamura number. With our interpretation we can add that even if the action of the simple game is unstable the Nakamura number provides a measure of instability: it is viewed as a stability index.

The second class for which we determine the stability index is that of maximal effectivity functions (Theorem 4.10). This is an important case in applications since the β -effectivity associated to a strategic game form is maximal. The index is determined by checking classical properties: regularity, superadditivity and subadditivity. Our method for the general case consists in extracting two appropriate effectivity functions from the local effectivity function, and to check whether they coincide. When this is the case, the local effectivity function is said to be exact. Again, in the class of maximal local effectivity functions, we can determine the stability index: by checking exactness and classical properties of effectivity functions (Theorem 4.16). It is remarkable that for unstable maximal effectivity functions and indeed for unstable maximal local effectivity functions, the stability index is always either 2 or 3.

The paper is organized as follows: In Section 2, local effectivity functions and related concepts are defined. Cycles are the main object of Section 3. Section 4 is devoted to the study of the stability index. The index is defined, motivated, interpreted and illustrated by examples. Its relationship to the Nakamura number is established in Subsection 4.2. The determination of the stability index of maximal effectivity functions is the object of Subsection 4.3 and that of maximal local effectivity functions is the object of Subsection 4.4. In Subsection 4.5, we deduce a classification of strategic game forms based on the stability index of the β -core and the exact core solutions. We conclude in Section 5.

2 The model

In this section we define a model of interaction that specifies the power distribution of a set of agents over some set of alternatives, with no explicit reference to any strategic mechanism that gives rise to that power. We shall see later (definition 2.5) how, starting from a strategic mechanism (i.e. a game form) one can derive an appropriate description of the power distribution induced by the strategies. The notions that we present in this section, have in common that only the *independent* power held by coalitions is represented. They are encompassed by the concept of local effectivity function. We shall see that the latter includes, as particular cases, effectivity functions and simple games.

2.1 Basic notations

Throughout this paper we shall consider a finite set N , the elements of which are called players or agents, and a finite set A , the elements of which are called alternatives or states. We make use of the following notational conventions: For any set D , we denote by $\mathcal{P}(D)$ the set of all subsets of D and by $\mathcal{P}_0(D) = \mathcal{P}(D) \setminus \{\emptyset\}$ the set of all non-empty subsets of D . Elements of $\mathcal{P}_0(N)$ are called *coalitions*. $N \setminus S$ is denoted S^c . Similarly if $B \in \mathcal{P}(A)$, $A \setminus B$ is denoted B^c . $L(A)$ will denote the set of all linear orders on A (that is all binary relations on A which are complete, transitive, and antisymmetric). $R \in L(A)$ will be interpreted as a preference relation on A . A preference profile (over A) is a map from N to $L(A)$, so that a preference profile is an element of $L(A)^N$. For every preference profile $R_N \in L(A)^N$ and $S \in \mathcal{P}_0(N)$ we put

$$P(a, S, R_N) = \{b \in A \mid b \neq a, b R^i a, \forall i \in S\}$$

(so that $P(a, S, R_N)$ consists of all the outcomes considered to be better preferred to a by all members of the coalition S), and $P^c(a, S, R_N) = A \setminus P(a, S, R_N)$.

2.2 Local effectivity functions

In the study of game form solvability, the idea that the power of a coalition may depend on the current state, arises naturally. This is the reason why a local effectivity function was first introduced in Abdou (1995), but only as an object related to a game form and an equilibrium concept (e.g. Nash or strong Nash). In this paper we shall work with an abstract coalitional form, where the power of coalitions depends on the current state¹.

Definition 2.1 A *local effectivity function* on (N, A) is a family $\mathcal{E} \equiv (\mathcal{E}[U], U \in \mathcal{P}_0(A))$ where for any $U \in \mathcal{P}_0(A)$, $\mathcal{E}[U] : \mathcal{P}(N) \rightarrow \mathcal{P}(\mathcal{P}_0(A))$ and such that the following conditions are satisfied:

- (i) $\mathcal{E}[U](S) = \emptyset$ if and only if $S = \emptyset$,
- (ii) $B \in \mathcal{E}[U](S), B \subset B' \Rightarrow B' \in \mathcal{E}[U](S)$,
- (iii) $U \subset V \Rightarrow \mathcal{E}[V](S) \subset \mathcal{E}[U](S)$.

The formula $B \in \mathcal{E}[U](S)$ is interpreted as follows: When the current state is in U , coalition S can adapt its response in order to realize some state in B . Let $R_N \in L(A)^N$. An alternative $a \in A$ is *dominated* at R_N if there exists $U \in \mathcal{P}_0(A)$, $S \in \mathcal{P}_0(N)$ such that $a \in U$ and $P(a, S, R_N) \in \mathcal{E}[U](S)$. The *core* of \mathcal{E} at R_N is the set of undominated alternatives. It is denoted $C(\mathcal{E}, R_N)$. \mathcal{E} is *stable* if $C(\mathcal{E}, R_N) \neq \emptyset$ for all $R_N \in L(A)^N$. We introduce a

¹The idea to introduce a coalitional power description that depends on the state goes back to Rosenthal (1972).

partial order on the set of all local effectivity functions on (N, A) as follows: $\mathcal{E} \preceq \mathcal{E}'$ if and only if: $\forall U \in \mathcal{P}_0(A), \forall S \in \mathcal{P}_0(N) : \mathcal{E}[U](S) \subset \mathcal{E}'[U](S)$.

Local effectivity functions that appear in the litterature may satisfy some additional properties: The first one is *monotonicity w.r.t. players*:

$$(m) \forall U \in \mathcal{P}_0(A), \forall S \in \mathcal{P}_0(N), \forall T \in \mathcal{P}_0(N) : S \subset T \Rightarrow \mathcal{E}[U](S) \subset \mathcal{E}[U](T),$$

The second, *Possibility of Non-Action*, is defined as follows:

$$(a) \forall U \in \mathcal{P}_0(A), \forall S \in \mathcal{P}_0(N) : U \in \mathcal{E}[U](S),$$

The third one is the *sheaf property*:

$$(s) \forall S \in \mathcal{P}_0(N), \forall U \in \mathcal{P}_0(A) : \mathcal{E}[U](S) = \cap_{a \in U} \mathcal{E}[\{a\}](S).$$

Although they may play a role in some circumstances, these properties are not needed for the most part of this study. In the following remark we show their impact on the core correspondence:

Remark 2.2 Let (x) be any of the properties $(m), (a), (s)$. Given any local effectivity function \mathcal{E} we denote by $\mathcal{E}^{(x)}$ the smallest (for \preceq) local effectivity function \mathcal{E}' that satisfies property (x) and such that $\mathcal{E} \preceq \mathcal{E}'$. If we note $\mathcal{E}^{(xy)}$ the result on \mathcal{E} of the operation (x) followed by the operation (y) , it is easy to see that $\mathcal{E}^{(xx)} = \mathcal{E}^{(x)}$ and $\mathcal{E}^{(xy)} = \mathcal{E}^{(yx)}$ ($x, y \in \{m, a, s\}$). Moreover for any $R_N \in L(A)^N$ and any $x \in \{m, a, s\}$ one has: $C(\mathcal{E}^{(x)}, R_N) = C(\mathcal{E}, R_N)$. One can prove that given two local effectivity functions \mathcal{E} and \mathcal{F} , $C(\mathcal{E}, R_N) = C(\mathcal{F}, R_N)$ for all $R_N \in L(A)^N$ if and only if $\mathcal{E}^{(mas)} = \mathcal{F}^{(mas)}$.

We now present some notions that appear in social choice theory and show how they can be viewed as particular cases of local effectivity functions. The following can be traced back to Moulin and Peleg (1982):

Definition 2.3 An *effectivity function* on (N, A) is a mapping $E : \mathcal{P}(N) \rightarrow \mathcal{P}(\mathcal{P}_0(A))$ such that :

- (i) $E(S) = \emptyset$ if and only if $S = \emptyset$,
- (ii) $B \in E(S), B \subset B' \Rightarrow B' \in E(S)$.

To any effectivity function E , we shall associate the local effectivity function defined by: $\mathcal{E}_E[U] = E$ for any $U \in \mathcal{P}_0(A)$. In this paper, an effectivity function E will be considered as a particular case of local effectivity function, via the identification of E to \mathcal{E}_E . In an effectivity function the power of a coalition is independent of the current state. An effectivity function is thus the analog of a cooperative game with abstract payoffs.

Our second notion is a generalization of simple games as defined in Nakamura (1975):

Definition 2.4 A *local simple game* on (N, A) is a collection $\mathcal{W} = (W_a, a \in A)$ where $W_a \subset \mathcal{P}_0(N)$, $(a \in A)$. When $W_a = W$ for all $a \in A$, we have a (standard) simple game (W, A) .

W_a is the set of winning coalitions at a . The interpretation of a local simple game is as follows: if the current state is a then any coalition in W_a has the power to react so that to reach any $b \in A$. For $U \in \mathcal{P}_0(A)$, put $\mathcal{W}[U] := \bigcap_{a \in U} W_a$. To any local simple game, we associate a local effectivity function that reflects the same power distribution. It is defined by $\mathcal{E}[U](S) = \mathcal{P}_0(A)$ if $S \in \mathcal{W}[U]$ and $\mathcal{E}[U](S) = \{A\}$ if $S \notin \mathcal{W}[U]$, $S \neq \emptyset$. In a local simple game, given some current state, a coalition is either totally powerful or totally powerless.

The third notion comes from strategic game theory. We consider a strategic game form $G = \langle (X_i)_{i \in N}, A, g \rangle$ where X_i is the *strategy set* of player i , $(i \in N)$ and $g : \prod_{i \in N} X_i \rightarrow A$ is the *outcome function*. We assume that g is onto. If $S \in \mathcal{P}_0(N)$ we denote by X_S the cartesian product $\prod_{i \in S} X_i$. If $x_N \in X_N$ we may also write $x_N = (x_S, x_{S^c})$.

Definition 2.5 Let G be a strategic game form. The *local effectivity function* \mathcal{E}^G associated to G is defined as follows: For $U \in \mathcal{P}_0(A)$, $\mathcal{E}^G[U](\emptyset) = \emptyset$ and for $S \in \mathcal{P}_0(N)$:

$$\mathcal{E}^G[U](S) = \{B \in \mathcal{P}_0(A) \mid \forall x_N \in g^{-1}(U), \exists y_S \in X_S : g(x_{S^c}, y_S) \in B\}$$

The β -effectivity function associated to G is defined by the formula: $E_\beta^G = \mathcal{E}^G[A]$.

The local effectivity function associated to a strategic game form was first introduced in Abdou (2000) in relation to strong Nash solvability. It satisfies properties (m), (a) and (s). Note that this notion is of reactive type (or β -type): given a current set of alternatives U , it tells whether a coalition S can oppose it, by threatening to achieve some other set of alternatives namely B .

3 Cycles

The notion of cycle is the basic object on which we shall build, in the next section, our definition of index. It can be viewed as a generalization of the famous Condorcet cycle. In the study of effectivity functions and their stability, two elementary cycles appeared naturally in Abdou (1982). General cycles were defined in Keiding (1985). Later on this notion has been extended to more general structures (Abdou and Keiding 2003). The following definition is an adaptation of this notion to local effectivity functions.

Definition 3.1 An r -tuple $((C_1, B_1, S_1), \dots, (C_r, B_r, S_r))$ where $r \geq 1$, $C_k \in \mathcal{P}_0(A)$, $B_k \in \mathcal{P}_0(A)$, $S_k \in \mathcal{P}_0(N)$ ($k = 1, \dots, r$) is an \mathcal{E} -configuration if:

(i) $B_k \in \mathcal{E}[C_k](S_k)$ ($k = 1, \dots, r$).

An \mathcal{E} -configuration $((C_1, B_1, S_1), \dots, (C_r, B_r, S_r))$ is a *cycle* if:

(ii) $\cup_{k=1}^r C_k = A$,

(iii) For any $\emptyset \neq J \subset \{1, \dots, r\}$ such that $\cap_{k \in J} S_k \neq \emptyset$, there exists $k \in J$ such that for all $l \in J$: $B_k \cap C_l = \emptyset$.

(C_1, \dots, C_r) is said to be the *basis* of the cycle and r its *length*. If (C_1, \dots, C_r) is a partition of A the cycle is said to be *strict*. \mathcal{E} is said to be *acyclic* if it has no cycle.

Condition (i) states that at any alternative in C_k , coalition S_k can achieve B_k , and therefore that coalition S_k would object against C_k if it were in the interest of its members to do so. Condition (ii) insures that all alternatives will be opposed. Condition (iii) is a combinatorial property that insures that the sequence of potential objections is somehow consistent. This consistency concerns only intersecting families of coalitions. Let I be a subset of $\{1, \dots, r\}$ with $\cap_{k \in I} S_k \neq \emptyset$. One can see that if condition (iii) is fulfilled for all non-empty subsets J of I , then there exists an order on I say $k_1, \dots, k_{|I|}$ such that $[(C_{k_1} \cup \dots \cup C_{k_j})] \cap [B_{k_j} \cup \dots \cup B_{k_{|I|}}] = \emptyset$ ($j = 1, \dots, |I|$). Conversely if such an order exists then condition (iii) is fulfilled for all non-empty subsets J of I . Using the notations of definition 3.1, some immediate consequences are in order:

(1) For any cycle: $C_k \cap B_k = \emptyset$ ($k = 1, \dots, r$). This is a consequence of condition (iii), where we take $J = \{r\}$.

(2) For any cycle: $\cap_{k=1}^r B_k = \emptyset$: This follows from (1) and condition (ii).

(3) For any cycle: $\cap_{k=1}^r S_k = \emptyset$: If not, then by condition (iii) there would exist $k \in \{1, \dots, r\}$ such that $B_k = \emptyset$, a contradiction.

(4) Any cycle has length ≥ 2 : Indeed, in view of (2) or (3) one has $r > 1$.

(5) Any cycle of length r gives rise to a strict cycle of length $\leq r$. This can be done as follows: Put $\tilde{C}_1 = C_1$ and $\tilde{C}_k = C_k \setminus \cup_{l=1}^{k-1} C_l$ ($2 \leq k \leq r$), and remove the indices k corresponding to empty \tilde{C}_k .

(6) Any cycle has length $\leq |A|$. This follows from (5) since the cardinality of a partition of A is less or equal to $|A|$.

Example 3.2 1) An \mathcal{E} -configuration $((C_1, B_1, S_1), (C_2, B_2, S_2))$ is a 2-cycle if and only if: $C_1 \cup C_2 = A$, $S_1 \cap S_2 = \emptyset$ and $B_k \cap C_k = \emptyset$ ($k = 1, 2$).

2) An \mathcal{E} -configuration $((C_1, B_1, S_1), \dots, (C_r, B_r, S_r))$ such that for all $k, l = 1, \dots, r$, $k \neq l$, $S_k \cap S_l = \emptyset$ is a cycle if and only if $\cup_{k=1}^r C_k = A$ and $C_k \cap B_k = \emptyset$ for all $k \in \{1, \dots, r\}$.

Let E be an effectivity function (definition 2.3). We define a *cycle* of E simply as a cycle of \mathcal{E}_E , where \mathcal{E}_E is the local effectivity function associated to E . We exhibit as an example two elementary types of cycles as they appeared in Abdou (1982):

Example 3.3 (a) Let E be an effectivity function (definition 2.3). An r -tuple $((B_1, S_1), \dots, (B_r, S_r))$ where $r \geq 2$, $S_k \in \mathcal{P}_0(N)$, $B_k \in E(S_k)$ ($k = 1, \dots, r$), $B_k \cap B_l = \emptyset$ ($k \neq l$) and $\bigcap_{k=1}^r S_k = \emptyset$ gives rise to a cycle of E . We have a basis by putting: $C_k = B_{k-1}$, $k = 2, \dots, r$ and $C_1 = A \setminus \bigcup_{k=2}^r C_k$.
(b) Let E be an effectivity function. An r -tuple $((B_1, S_1), \dots, (B_r, S_r))$ where $r \geq 2$, $S_k \in \mathcal{P}_0(N)$, $B_k \in E(S_k)$ ($k = 1, \dots, r$), $S_k \cap S_l = \emptyset$ ($k \neq l$) and $\bigcap_{k=1}^r B_k = \emptyset$ gives rise to a cycle of E . By putting $C_k = B_k^c$ ($k = 1, \dots, r$) (and removing those indices k with $B_k^c = \emptyset$) we have a basis.

It was asserted earlier (remark 2.2) that if \mathcal{E} is any local effectivity function and if (x) is any of the properties (m) , (a) , (s) then $\mathcal{E}^{(x)}$ and \mathcal{E} have the same core correspondence. As regards cycles, we have the following :

Remark 3.4

- 1) $\mathcal{E}^{(a)}$ and \mathcal{E} have the same cycles,
- 2) To any cycle of $\mathcal{E}^{(m)}$ corresponds some cycle of \mathcal{E} with the same basis (hence the same length), and vice versa,
- 3) To any cycle of \mathcal{E} corresponds some cycle of $\mathcal{E}^{(s)}$ (not necessarily with the same length), and vice versa.

We end this section by stating the main result that justifies the introduction of cycles. In the case of effectivity functions, it was first proved by Keiding (1985) (see also Abdou and Keiding, 1991, Theorem 5.3). For a more general result that covers the case of local effectivity functions we refer to Abdou and Keiding (2003), Theorem 6.

Theorem 3.5 *A local effectivity function \mathcal{E} is stable if and only if it is acyclic.*

4 Instability and the stability index

This section is devoted to the study of unstable local effectivity functions. It would be interesting to have a typology of configurations that generate instability. Since instability has to do with the existence of cycles (Theorem 3.5), it is clear that such a typology may be founded on the set of cycles. As a first step toward this end, we shall provide an index that induces a classification of unstable local effectivity functions. Obviously, no single

integer can pretend to exhaust all features of instability. However, in order that such an index be relevant some requirements seem to be natural:

- (1) An index is a real number (or an integer) defined on the class of local effectivity functions. It is relevant for the local effectivity function not for the derived game contingent on some profile. The idea is to have an *a priori* measure for the stability of power distribution allowed by some institutional mechanism (e.g. written or unwritten political rules of government or justice), rather than a measure of the robustness of some concrete situation.
- (2) If \mathcal{E} and \mathcal{F} are local effectivity functions with the same players and the same alternatives and if all cycles in \mathcal{E} are cycles in \mathcal{F} then the index of \mathcal{E} must be greater or equal to the index of \mathcal{F} . The index is thus an increasing function of stability. If the latter is deemed a desirable property, a mechanism with a higher index will be valued more than one with a lower index.

Instability occurs with the emergence of some \mathcal{E} -configuration that has properties (ii) and (iii) of definition 3.1. This involves potentially the formation of a coalition structure S_1, \dots, S_r , and for $k = 1 \dots, r$, the devising of a coordinated objection B_k within the members of S_k , that may oppose C_k . Since the formation of coalitions and coordination of actions are rather difficult and costly, we implicitly postulate that smaller \mathcal{E} -configurations are more likely to surface than larger ones². Although cycles of the same length may be very different in structure (see example 3.3 where cycle (b) seems to be “simpler” than cycle (a)), we put forward the idea that a good index of stability of power distribution has to do with the shortest cycles. In accordance with this idea we introduce the following:

Definition 4.1 The *stability index* of \mathcal{E} , denoted $\sigma(\mathcal{E})$, is the minimal length of a cycle in \mathcal{E} . $\sigma(\mathcal{E})$ is set to $+\infty$ if \mathcal{E} is acyclic.

Clearly our index satisfies the above requirements (1) and (2). Moreover it is integer-valued, and for an unstable local effectivity function it takes values between 2 and $|A|$.

Grouping alternatives plays an important role in the definition of the index. If we were concerned only by characterizing stability we could have restricted the study to \mathcal{E} -configurations $\mathcal{C} \equiv ((C_1, B_1, S_1), \dots, (C_r, B_r, S_r))$ with $|C_k| = 1$ ($k = 1, \dots, r$). A cycle of this form is necessarily of length $|A|$. Under such a restriction Theorem 3.5 would remain true but definition 4.1 would be useless. In a cycle C_k plays two roles: it is a set of alternatives where S_k can implement its objection B_k , and the package that will be

²I am indebted to an anonymous referee for this motivation concerning our index and for other suggestions that improved the content of this paper

opposed by S_k . The following paragraph will give an insight into the role of the sheaf property in packaging alternatives.

Let \mathcal{E} be a local effectivity function that satisfies the sheaf property (s). Using notations of definition 3.1, let $\mathcal{C} \equiv ((C_1, B_1, S_1), \dots, (C_r, B_r, S_r))$ be an \mathcal{E} -configuration. \mathcal{C} is said to be redundant if there exist indices k, k' $k \neq k'$, such that $S_k = S_{k'}$ and $B_k = B_{k'}$. By removing k' from the set of indices and replacing C_k by $C_k \cup C_{k'}$, one obtains a shorter \mathcal{E} -configuration, say \mathcal{C}' . If \mathcal{C} is a cycle of \mathcal{E} , then so is \mathcal{C}' . It follows that the index of \mathcal{E} is the length of a non redundant cycle. If \mathcal{E} does not satisfy the sheaf property, it may well happen that \mathcal{C} is a cycle but not \mathcal{C}' (see Remark 3.4 (3)). If \mathcal{C} is of minimal length, then the index of \mathcal{E} is the length of \mathcal{C} , a redundant cycle. We see here the role played by grouping alternatives: for some reason inherent in the power structure, coalition S_k has to form twice in order to devise the same objection $B_{k'} = B_k$ that will be opposed to C_k and $C_{k'}$. If we replace \mathcal{E} by its sheaf cover $\mathcal{E}^{(s)}$, then the index will drop at least by 1. The following example illustrates this fact:

Example 4.2 $N = \{1, 2\}$, $A = \{a, b, c\}$,

$$\mathcal{E}[U](\{1\}) = \mathcal{E}[U](\{2\}) = \mathcal{P}_0(A) \text{ if } |U| = 1,$$

$$\mathcal{E}[U](\{1\}) = \mathcal{E}[U](\{2\}) = \{A\} \text{ if } |U| \geq 2,$$

$$\mathcal{E}[U](N) = \mathcal{P}_0(A) \text{ if } U \neq \emptyset.$$

Any 2-cycle of \mathcal{E} , say $((C_1, B_1, S_1), (C_2, B_2, S_2))$ (example 3.2) is such that $S_1 \cap S_2 = \emptyset$. W.l.o.g we shall take $S_1 = \{1\}$, $S_2 = \{2\}$. Since $C_1 \cup C_2 = A$, either $|C_1| \geq 2$ or $|C_2| \geq 2$. It follows that either $B_1 = A$ or $B_2 = A$. This contradicts $B_1 \cap C_1 = \emptyset$ and $B_2 \cap C_2 = \emptyset$. It follows that \mathcal{E} has no cycle of length 2. On the other hand, one can consider the sheaf cover $\mathcal{E}^{(s)}$ of \mathcal{E} :

$$\mathcal{E}^{(s)}[U](\{1\}) = \mathcal{E}^{(s)}[U](\{2\}) = \mathcal{E}[U](N) = \mathcal{P}_0(A) \text{ for all } U \neq \emptyset,$$

so that we have a 2-cycle of $\mathcal{E}^{(s)}$ if we take: $C_1 = \{a, b\}$, $B_1 = \{c\}$, $S_1 = \{1\}$, $C_2 = \{c\}$, $B_2 = \{a\}$, $S_2 = \{2\}$.

We conclude that \mathcal{E} is an unstable local effectivity function such that $\sigma(\mathcal{E}) = 3$ and $\sigma(\mathcal{E}^{(s)}) = 2$.

The following subsection will clarify further the index definition via the introduction of merger of alternatives.

4.1 Merging alternatives

Let $f : A \rightarrow A'$ be a map, where A' is an arbitrary finite set. Let \mathcal{E} be a local effectivity function on (N, A) . We define the image \mathcal{E}^f of \mathcal{E} by f as the local effectivity function on (N, A') where, for any $U' \in \mathcal{P}_0(A')$:

$$\mathcal{E}^f[U'](S) = \{B' \in \mathcal{P}_0(A') \mid f^{-1}(B') \in \mathcal{E}[(f^{-1}(U'))](S)\}$$

Lemma 4.3 *For any local effectivity function \mathcal{E} and any $f : A \rightarrow A'$ one has $\sigma(\mathcal{E}) \leq \sigma(\mathcal{E}^f)$*

Proof. The r -tuple $((C'_1, B'_1, S_1), \dots, (C'_r, B'_r, S_r))$ is a cycle of \mathcal{E}^f if and only if $((f^{-1}(C'_1), f^{-1}(B'_1), S_1), \dots, (f^{-1}(C'_r), f^{-1}(B'_r), S_r))$ is a cycle of \mathcal{E} . \square

Now let $((C_1, B_1, S_1), \dots, (C_r, B_r, S_r))$ be a cycle of \mathcal{E} based on the partition (C_1, \dots, C_r) . Let A' be some set with r elements $A' := \{u_1, \dots, u_r\}$ and let $f : A \rightarrow A'$ be defined by $f(a) = u_k$ if $a \in C_k$. Put $B'_k := f(B_k)$ $k = 1, \dots, r$. For any $k, l \in \{1, \dots, r\}$ one has $C_k \cap B_l = \emptyset$ if and only if $\{u_k\} \cap f(B_l) = \emptyset$. It follows that $((\{u_1\}, B'_1, S_1), \dots, (\{u_r\}, B'_r, S_r))$ is a cycle of \mathcal{E}^f based on the partition $(\{u_1\}, \dots, \{u_r\})$. Therefore we have the following characterization:

Theorem 4.4 *The stability index of a local effectivity function \mathcal{E} is the smallest integer s for which the following property holds:*

There exists a surjection $f : A \rightarrow \{1, \dots, s\}$ such that \mathcal{E}^f is unstable.

Proof. Let s be the number defined in the claim. Then for some $f : A \rightarrow \{1, \dots, s\}$, \mathcal{E}^f has a cycle. By Lemma 4.3 $\sigma(\mathcal{E}) \leq \sigma(\mathcal{E}^f) \leq s$. Since there exists a cycle of length $\sigma(\mathcal{E})$ in \mathcal{E} , using the argument that precedes the statement, there exists a surjection $f : A \rightarrow \{1, \dots, \sigma(\mathcal{E})\}$ such that \mathcal{E}^f is unstable, so that $s \leq \sigma(\mathcal{E})$. It follows that $s = \sigma(\mathcal{E})$. \square

This characterization allows for an interpretation of the stability index. Assume that a local effectivity function is unstable with a stability index σ , then merging some alternatives results in a transformation of the local effectivity function in a way that respects the power distribution. This is the interpretation of the operation $\mathcal{E} \rightarrow \mathcal{E}^f$. This transformation may occur, for instance, when the agents cease to distinguish between two previously distinct alternatives. If the cardinality of the new set is inferior to σ , then the new local effectivity function is stable. In order to produce a deadlock, players have to show some level of sophistication. Therefore if the legislator seeks stability and if a stable mechanism is not available, the more acute is the perception of alternatives, the higher index must be recommended.

Example 4.5 If the power distribution is such that $\sigma = 2$, then instability takes a particularly simple form: alternatives can be partitioned into two aggregates, or two major issues, on which the society is split, and the power of coalitions allowed by the rules is such that both issues can be opposed and neither one can be forced. Many countries present the property of being politically split over two main issues. History and geography are accountable for this bipolarity. The main issues can be of socioeconomic type, or of ethnic or religious type. Almost all Western countries are divided between left and right, conservative and liberal, democrat and republican. Many Middle-Eastern societies are split into pro-Western and anti-Western coalitions. In

such a context, whatever are the original alternatives, the perception of political issues may be represented by $f : A \rightarrow A'$, where $|A'| = 2$. Bipolarity does not necessarily translate into instability. Indeed the stability index is determined by the power distribution (constitution or unwritten rules of government) and in most Western countries (“democracies”), where governance is based on a written constitution and elections, lawmakers strive to define constitutions that avoid instability generated by any bipolar split: the index is at least 3. By contrast some countries in the Middle-East did experience recently this type of instability. Pro-Western coalitions formally became the ruling power but they could not force any outcome. The anti-Western coalition itself could oppose any outcome but could force none: Political analysts express this situation by the vocable “stalemate” or “deadlock”: the index is presumably 2.

When σ is high, some configurations leading to instability may be combinatorially complex and in order to produce them, the society must have rather complex views.

Example 4.6 Some countries, though immune to bipolar deadlocks, could experience more sophisticated types of instability. Many parties with distinct political agendas exist simultaneously. Legal institutions work correctly and choose some ruling coalition with some program. The ruling coalition includes two or more parties who agree on some government issues. But the exercise of power becomes impossible when there is a disagreement within the coalition over the implementation of some new issue. Some party in the opposition proposes an alliance to some component of the ruling coalition. As a result the ruling coalition will eventually be overthrown, and new elections will be held. This scenario may repeat itself. The social context can be described by some map f with range ≥ 3 . Lawmakers designed institutions that are immune against bipolar instability so that the index is presumably 3; but the degree of sophistication of the society is larger so that instability may occur. It is important from the point of view of political science to distinguish between this type of instability and the bipolar stalemate.

The question of whether the probability (for instance when the preferences are assumed to be uniformly distributed) of reaching a cyclic configuration is related to the stability index remains open and is not addressed in this paper. However this relation, if ever it exists, is not straightforward as can be seen from the fact that two local effectivity functions may have the same core correspondence but not the same index ! In view of remark 3.4 and the discussion following definition 4.1, it may occur that for some local effectivity function \mathcal{E} , the index of $\mathcal{E}^{(s)}$ is strictly smaller than that of \mathcal{E} . However since \mathcal{E} and $\mathcal{E}^{(s)}$ have the same core correspondence, it follows that

the set of profiles that lead to an empty core is the same for \mathcal{E} and $\mathcal{E}^{(s)}$. Any index based on the probability of profiles with empty core would provide equal values for \mathcal{E} and $\mathcal{E}^{(s)}$. Therefore such an index conveys a different kind of information about stability than the one given by our index. In the rest of this section we shall compute or simply localize the stability index of some subclasses of local effectivity functions. We start by simple games.

4.2 Stability Index and the Nakamura Number

In the case of simple games there is a relationship between the stability index and the Nakamura number as defined in Nakamura (1979). This relationship casts light on the stability index and shows that, in a sense, the stability index may be viewed as the analog of the Nakamura number for local effectivity functions. Let W be a set of winning coalitions on N . (S_1, \dots, S_r) where $S_k \in W$ ($k = 1, \dots, r$) is said to be a *non intersecting family* of W if $\cap_{k=1}^r S_k = \emptyset$. The *Nakamura Number* of W , denoted $\nu(W)$, is defined as the minimum length of a non intersecting family. If W has no non intersecting family, then we set $\nu(W) = +\infty$. Let \mathcal{W} be a local simple game on (N, A) as in definition 2.4. A $2r$ -tuple $(U_1, S_1, \dots, U_r, S_r)$ where $U_k \in \mathcal{P}_0(A)$, $S_k \in \mathcal{W}[U_k]$ ($k = 1, \dots, r$) is said to be a *cycle* of \mathcal{W} if (U_1, \dots, U_r) is a partition of A and $\cap_{k=1}^r S_k = \emptyset$. The natural number r is the *length* of the cycle. We recall that the local effectivity function associated to \mathcal{W} (resp. (W, A)) is $\mathcal{E}^{\mathcal{W}}$ (resp. $\mathcal{E}^{W,A}$). Let $\sigma(\mathcal{W})$ (resp. $\sigma(W, A)$) denote the stability index $\sigma(\mathcal{E}^{\mathcal{W}})$ (resp. $\sigma(\mathcal{E}^{W,A})$). One has the following:

Lemma 4.7 *Any cycle of \mathcal{W} gives rise to some strict cycle of $\mathcal{E}^{\mathcal{W}}$ of the same length and vice versa.*

Proof. Let $(U_1, S_1, \dots, U_r, S_r)$ be a cycle in \mathcal{W} . Indices are taken in $\mathbb{Z}/r\mathbb{Z}$. Let $B_k := U_{k+1}$ ($k \in \mathbb{Z}/r\mathbb{Z}$). We claim that $((U_1, B_1, S_1), \dots, (U_r, B_r, S_r))$ is a strict cycle of $\mathcal{E}^{\mathcal{W}}$: In order to prove condition (iii) of definition 3.1 we remark that if J is such that $\cap_{j \in J} S_j \neq \emptyset$ then $J \neq \{1, \dots, r\}$ and we can choose any $k \in J$ such that $k+1 \notin J$. Conversely any strict cycle $((U_1, B_1, S_1), \dots, (U_r, B_r, S_r))$ in $\mathcal{E}^{\mathcal{W}}$ is such that (U_1, \dots, U_r) is a partition of A and $\cap_{k=1}^r S_k = \emptyset$. \square

Corollary 4.8 *For any simple game (W, A) one has :*

$$\sigma(W, A) = \nu(W) \quad \text{if } \nu(W) \leq |A| \quad (1)$$

$$= +\infty \quad \text{if } \nu(W) > |A| \quad (2)$$

In particular (W, A) is stable if and only if $\nu(W) > |A|$.

Proof. Let $\nu := \nu(W)$ and $\sigma := \sigma(W, A)$. If $\nu \leq |A|$, let (S_1, \dots, S_ν) be a non intersecting family of W . Let U_1, \dots, U_ν be any partition of A , then

$(U_1, S_1, \dots, U_\nu, S_\nu)$ is a cycle of (W, A) . It follows that $\sigma \leq \nu$. Conversely any cycle $(U_1, S_1, \dots, U_\sigma, S_\sigma)$ implies that (S_1, \dots, S_σ) is an intersecting family, so that $\nu \leq \sigma$. If $\nu > |A|$ then there can be no cycle in (W, A) since there is no partition of A of length ν . \square

An alternative way to express the relationship between $\sigma(W, A)$ and $\nu(W)$ is as follows:

$$\nu(W) = \sigma(W, A) \quad \text{if } \sigma(W, A) < +\infty \quad (3)$$

$$> |A| \quad \text{if } \sigma(W, A) = +\infty \quad (4)$$

4.3 Stability index of Effectivity functions

It is possible to refine our knowledge of the stability index for some classes of effectivity functions. For that purpose we recall some properties that appear in the study of stability. As will be seen in this subsection they have a fundamental role in the determination of the stability index. An effectivity function E is said to be:

monotonic w.r.t. players if for all $S, T \in \mathcal{P}_0(N)$,

$$S \subset T \Rightarrow E(S) \subset E(T), \quad (5)$$

regular if for all $S_1 \in \mathcal{P}_0(N), S_2 \in \mathcal{P}_0(N)$,

$$S_1 \cap S_2 = \emptyset, B_1 \in E(S_1), B_2 \in E(S_2) \Rightarrow B_1 \cap B_2 \neq \emptyset, \quad (6)$$

maximal if for all $S \in \mathcal{P}_0(N), B \in \mathcal{P}_0(A)$,

$$B^c \notin E(S^c) \Rightarrow B \in E(S), \quad (7)$$

superadditive if for all $S_1 \in \mathcal{P}_0(N), S_2 \in \mathcal{P}_0(N)$,

$$S_1 \cap S_2 = \emptyset, B_1 \in E(S_1), B_2 \in E(S_2) \Rightarrow B_1 \cap B_2 \in E(S_1 \cup S_2), \quad (8)$$

subadditive if for all $S_1 \in \mathcal{P}_0(N), S_2 \in \mathcal{P}_0(N)$,

$$B_1 \cap B_2 = \emptyset, B_1 \in E(S_1), B_2 \in E(S_2) \Rightarrow B_1 \cup B_2 \in E(S_1 \cap S_2). \quad (9)$$

The β -Effectivity function E_β^G associated to a strategic game form G (definition 2.5) satisfies monotonicity w.r.t. players and maximality. Therefore studying the maximal case is particularly important for applications. Superadditivity and subadditivity play a crucial role in the description of the structure of cycles. If an effectivity function is subadditive then it has no cycle of type (a) of example 3.3; if it is superadditive, then it has no cycle of type (b). It is interesting to note that in the case of maximal effectivity functions, the absence of such cycles (types (a) or (b)) is equivalent to stability. The following clear cut result that can be deduced from Abdou (1982) and Peleg (1984) (Theorem 6.A.9) is reproduced here for future use:

Theorem 4.9 *Let E be a maximal effectivity function. The following are equivalent:*

- (i) E is stable,
- (ii) E is superadditive and subadditive,
- (iii) E has no cycles of type (a) or (b) of example 3.3.

As regards the stability index, we can establish, relying on Theorem 4.9, the following:

Theorem 4.10 *Let E be an effectivity function. Then:*

- (i) $\sigma(E) > 2$ if and only if E is regular.
- (ii) Assume that E is maximal. Then $\sigma(E) \in \{2, 3, +\infty\}$.

Proof. (i) Let $((C_1, B_1, S_1), (C_2, B_2, S_2))$ be a 2-cycle. Then $S_1 \cap S_2 = \emptyset$, $C_1 \cup C_2 = A$, $B_i \in E(S_i)$, $B_i \subset C_i^c$ ($i = 1, 2$), so that $B_1 \cap B_2 \subset C_1^c \cap C_2^c = \emptyset$. This contradicts regularity. Conversely if $((S_1, B_1), (S_2, B_2))$ is such that $B_i \in E(S_i)$ ($i = 1, 2$), $S_1 \cap S_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$ then $((B_1^c, B_1, S_1), (B_2^c, B_2, S_2))$ is 2-cycle.

(ii) In view of Theorem 4.9, a maximal effectivity function is stable if and only if it is subadditive and superadditive. In that case $\sigma(E) = +\infty$. If E is not superadditive then there exists $S_1, S_2 \in \mathcal{P}_0(N)$, $B_1, B_2 \in \mathcal{P}_0(A)$ such that $S_1 \cap S_2 = \emptyset$, $B_1 \in E(S_1)$, $B_2 \in E(S_2)$ and $B_1 \cap B_2 \notin E(S_1 \cup S_2)$. Put $S_3 = (S_1 \cup S_2)^c$, $B_3 = (B_1 \cap B_2)^c$. By maximality $B_3 \in E(S_3)$ so that $((B_1^c, B_1, S_1), (B_2^c, B_2, S_2), (B_3^c, B_3, S_3))$ is a cycle. A similar proof can be done if E is not subadditive. Therefore $\sigma(E) \leq 3$. \square

Since most effectivity functions met in applications are the β -effectivity of some game form, and since those are maximal, theorem 4.10 can be viewed as a rather negative result.

4.4 Stability Index of Local Effectivity Functions

In order to obtain some precise indications on the stability index of a local effectivity function, we shall define some simpler objects that it induces and that play a role in its stability. In his study of strong Nash solvability, together with the local effectivity functions associated to a game form, Abdou (1995) introduced a property called exactness. The latter generalizes “exact maximality”, a property found in Li (1991). Here we extend this definition to general local effectivity functions. Starting from a local effectivity function \mathcal{E} we define two simpler objects:

The *global effectivity function derived from \mathcal{E}* is the mapping $E_0 : \mathcal{P}(N) \rightarrow \mathcal{P}(\mathcal{P}_0(A))$ such that all $S \in \mathcal{P}(N)$: $E_0(S) := \mathcal{E}[A](S)$.

The *exact effectivity function derived from \mathcal{E}* is the mapping $E_\xi : \mathcal{P}(N) \rightarrow \mathcal{P}(\mathcal{P}_0(A))$ such that $E_\xi(\emptyset) = \emptyset$ and for $S \in \mathcal{P}_0(S)$, by:

$$E_\xi(S) := \{B \in \mathcal{P}_0(A) \mid B = A \text{ or } \exists a \notin B, B \in \mathcal{E}[\{a\}](S)\} \quad (10)$$

Note that properties (i) and (ii) of definition 2.3 are satisfied by E_0 , whereas E_ξ does not necessarily satisfy property (ii) (monotonicity w.r.t. alternatives) so that it is not an “effectivity function”. In general it is clear from the definitions that for all $S \in \mathcal{P}_0(N)$, $E_0(S) \subset E_\xi(S)$.

Definition 4.11 A local effectivity function \mathcal{E} is said to be *exact* if $E_0 = E_\xi$. A local effectivity function \mathcal{E} is said to be *maximal* if E_0 is maximal.

Note that exactness conveys some information only in the framework of local effectivity functions. In the subclass of effectivity functions, that is, in the case where \mathcal{E} is independent of U , it is always true that $E_0 = E_\xi = \mathcal{E}[U]$ (for any $U \in \mathcal{P}_0(N)$). Exactness together with maximality will shed light on the structure of local effectivity functions with respect to the stability/instability problem. In what follows we investigate the impact of exactness on the core correspondence and cycles of \mathcal{E} . Clearly $C(\mathcal{E}, R_N) \subset C(E_0, R_N)$ for all $R_N \in L(A)^N$ and any cycle of E_0 is a cycle of \mathcal{E} . In addition one has:

Proposition 4.12 *If \mathcal{E} is exact, then:*

- (i) $C(E_0, R_N) = C(\mathcal{E}, R_N)$ for all $R_N \in L(A)^N$,
- (ii) \mathcal{E} and E_0 have the same cycles.

Proof. (i) Let $R_N \in L(A)^N$. If a is dominated in \mathcal{E} at R_N , then for some $S \in \mathcal{P}_0(N)$, $P(a, S, R_N) \in \mathcal{E}[\{a\}](S)$. Since $a \notin P(a, S, R_N)$, one has $P(a, S, R_N) \in E_\xi(S) = E_0(S)$ so that a is dominated in E_0 . This proves $C(E_0, R_N) \subset C(\mathcal{E}, R_N)$. Since the opposite inclusion is always true, we have proved (i).

(ii) If the r -tuple $((C_1 B_1, S_1), \dots, (C_r, B_r, S_r))$ is a cycle in \mathcal{E} , then in particular $C_k \cap B_k = \emptyset$, $C_k \neq \emptyset$ and since $B_k \in \mathcal{E}[C_k](S_k)$ it follows that $B_k \in E_\xi(S_k) = E_0(S_k)$, so that this r -tuple is also a cycle in E_0 . Since conversely, in all cases, any cycle in E_0 is a cycle in \mathcal{E} , we have proved (ii). \square

Lemma 4.13 *Assume that \mathcal{E} is maximal and not exact. Then:*

- (i) *there exists $R_N \in L(A)^N$ such that $C(\mathcal{E}, R_N) = \emptyset$ and if further E_0 is monotonic w.r.t. players we have in addition $|C(E_0, R_N)| = 1$.*
- (ii) *there exists a cycle of length at most 3 in \mathcal{E} , that is not a cycle in E_0 .*

Proof. (i) Let $S \in P_0(N)$, $B \in P_0(N)$ such that $B \in E_\xi(S)$ and $B \notin E_0(S)$. Then there exists $a \in B^c$ such that $B \in \mathcal{E}[\{a\}](S)$ and $B^c \in E_0(S^c)$. Define a profile R_N such that: for $i \in S$, $B R^i \{a\} R^i B^c \setminus \{a\}$ and for $i \in S^c$, $\{a\} R^i B^c \setminus \{a\} R^i B$. If $b \in B$ then $P(b, S^c, R_N) \supset B^c \in E_0(S^c)$, so that b is dominated in E_0 . If $b \in B \setminus \{a\}$, then $P(b, N, R_N) \supset \{a\}$. By maximality of E_0 , $E_0(N) = \mathcal{P}_0(A)$, so that b is dominated in E_0 . Therefore one has $C(E_0, R_N) \subset \{a\}$. Now $P(a, S, R_N) = B \in \mathcal{E}[\{a\}](S)$ implies that a is dominated in \mathcal{E} , and since $C(\mathcal{E}, R_N) \subset C(E_0, R_N)$ we have $C(\mathcal{E}, R_N) = \emptyset$. If moreover E_0 is monotonic w.r.t. players, then for $T \in \mathcal{P}_0(N)$, $T \subset S$ we have $P(a, T, R_N) = B \notin E_0(T)$ and for $T \cap S^c \neq \emptyset$ we have $P(a, T, R_N) = \emptyset$ so that a is not dominated in E_0 . We conclude that $C(E_0, R_N) = \{a\}$.

(ii) Put $S_1 = S, S_2 = S^c, S_3 = N$, $B_1 = B, B_2 = B^c, B_3 = \{a\}$, $C_1 = \{a\}, C_2 = B, C_3 = B^c \setminus \{a\}$. In \mathcal{E} this defines a 3-cycle if $C_3 \neq \emptyset$, and a 2-cycle if $C_3 = \emptyset$, that is not a cycle in E_0 .

As an immediate consequence of Lemma 4.13 we have the following proposition that provides, when E_0 is maximal, a partial converse to Proposition 4.12:

Proposition 4.14 *Assume that \mathcal{E} is maximal.*

(i) *If E_0 is monotonic w.r.t. players then \mathcal{E} is exact if and only if $C(E_0, R_N) = C(\mathcal{E}, R_N)$ for all $R_N \in L(A)^N$.*

(ii) *\mathcal{E} is exact if and only if E_0 and \mathcal{E} have the same cycles.*

In what follows, we summarize the main results concerning stability of local effectivity functions. This can be viewed as a generalization to the local effectivity functions, of Theorem 3.11 of Abdou (2000) where only objects derived from a game form were considered:

Theorem 4.15 *Assume that \mathcal{E} is maximal. Then the following are equivalent:*

(i) *\mathcal{E} is stable*

(ii) *\mathcal{E} is exact and E_0 is stable.*

(iii) *\mathcal{E} is exact and E_0 is superadditive and subadditive.*

Moreover in that case $C(E_0, R_N) = C(\mathcal{E}, R_N)$ for all $R_N \in L(A)^N$.

Proof. If \mathcal{E} is stable then E_0 is stable and by the first part of assertion (i) of Lemma 4.13, \mathcal{E} is exact. Conversely if \mathcal{E} is exact, then $C(E_0, R_N) = C(\mathcal{E}, R_N)$ for all $R_N \in L(A)^N$ by Lemma 4.12 assertion (i), and if moreover E_0 is stable it follows that \mathcal{E} is stable. This establishes the equivalence between (i) and (ii). The equivalence with (iii) follows from Theorem 4.9. \square

As regards instability one can obtain precise indications on the index. For that purpose we assert the following:

Theorem 4.16 (i) $\sigma(\mathcal{E}) = 2$ if E_0 is not regular.

(ii) Assume that \mathcal{E} is maximal. Then $\sigma(\mathcal{E}) \in \{2, 3, +\infty\}$

Proof. If E_0 is not regular then $2 \leq \sigma(\mathcal{E}) \leq \sigma(E_0) = 2$, we conclude that $\sigma(\mathcal{E}) = 2$. This proves (i). In order to prove (ii) assume that E_0 is maximal. We consider first the case where \mathcal{E} is not stable. Two subcases are possible. In the first subcase E_0 is stable, then by Theorem 4.15 \mathcal{E} is not exact and by Lemma 4.13 assertion (ii) \mathcal{E} has a cycle of length ≤ 3 , so that $\sigma(\mathcal{E}) \leq 3$. In the second subcase E_0 is not stable then by Theorem 4.10 $\sigma(E_0) \leq 3$ and since $\sigma(\mathcal{E}) \leq \sigma(E_0)$ we conclude again that $\sigma(\mathcal{E}) \leq 3$. Since in the case where \mathcal{E} is stable $\sigma(\mathcal{E}) = +\infty$, we have proved (ii). \square

Remark 4.17 If \mathcal{E} satisfies the non-action property (a) and the sheaf property (s) (see subsection 2.2) then (i) in 4.16 can be improved to read as follows: $\sigma(\mathcal{E}) = 2$ if and only if E_0 is not regular: If $((C_1, B_1, S_1), (C_2, B_2, S_2))$ is a 2-cycle, then $C_1 \cup C_2 = A$, $S_1 \cap S_2 = \emptyset$, $C_1 \subset B_1^c$ and $C_2 \subset B_2^c$, $B_1 \in \mathcal{E}[C_1](S_1)$, $B_2 \in \mathcal{E}[C_2](S_2)$. It follows that $C_1^c \in \mathcal{E}[C_1](S_1)$ and $C_2^c \in \mathcal{E}[C_2](S_2)$ and in view of properties (a) and (s), $C_1^c \in \mathcal{E}[A](S_1)$ and $C_2^c \in \mathcal{E}[A](S_2)$. Since $\mathcal{E}[A] = E_0$, $S_1 \cap S_2 = \emptyset$ and $C_1^c \cap C_2^c = \emptyset$ we deduce that E_0 is not regular.

As regards applications Theorem 4.16 can also be viewed as a rather negative result. Moreover, by the same theorem, the two types of instability described in examples 4.5 and 4.6 appear as paradigmatic.

4.5 Stability Index of strategic game forms

Let G be a strategic game form. For any preference profile, two coalitional solutions may be considered: the β -core and the exact core of G . Those are respectively the core of E_β^G and \mathcal{E}_β^G (see Abdou (2000)). We denote by $\sigma_0(G)$ and $\sigma_1(G)$ the stability index of E_β^G and \mathcal{E}_β^G respectively. G is said to be superadditive (resp. subadditive) if E_β^G is superadditive. G is said to be tight if E_β^G is regular. G is said to be exact if \mathcal{E}_β^G is exact. It is easy to see that, for any game form G , E_β^G is maximal, and if G is tight then G is superadditive. We therefore have a nice classification of strategic game forms:

Proposition 4.18 Let G be a game form. Then:

(i) $\sigma_0(G) = \sigma_1(G) = 2$ if and only if G is not tight,

(ii) $\sigma_0(G) = \sigma_1(G) = 3$ if and only if G is tight and not subadditive,

(iii) $\sigma_0(G) = +\infty$, $\sigma_1(G) = 3$ if and only if G is tight, subadditive and not exact,

(iii) $\sigma_0(G) = \sigma_1(G) = +\infty$ if and only if G is tight subadditive and exact.

We recall that a game form $G = (X_1, \dots, X_n, A, g)$ is said to be *rectangular* if for any $a \in A$, $g^{-1}(a) = \prod_{i=1}^n Y_i$, for some $Y_i \subset X_i$, ($i = 1, \dots, n$) (Gurvich (1978, 1989) and Abdou (1998, 2000)). In particular the normal form associated to any finite extensive game form where final nodes carry distinct outcomes (such extensive forms are called *free*), is rectangular. Exactness is a very discriminating property for the class of rectangular game forms. Indeed, in view of Theorem 4.7 of Abdou (2000), any exact rectangular game form G is essentially a one-player game form. The index $\sigma_1(G)$ for rectangular game forms can thus be determined with good precision:

Corollary 4.19 *Let G be a rectangular game form. Then there are three exclusive cases: (i) $\sigma_1(G) = 2$ if and only if G is not tight, (ii) $\sigma_1(G) = +\infty$ if and only if G is exact, (iii) $\sigma_1(G) = 3$ if and only if G is tight and not exact.*

Proof: By the remark that precedes our statement, it follows that \mathcal{E}_β^G is stable if and only if G is exact. If G is not exact, then either G is not tight and in this case $\sigma_1(G) = 2$, or it is tight and by 4.18 (ii) (iii), $\sigma_1(G) = 3$. \square

Remark 4.20 None of the cases of the corollary is empty. This can be concluded from Abdou (1998) or Gurvich (1978, 1989): A free extensive game form is strategically equivalent to some perfect information free extensive game form if and only if it is tight. It follows that any free extensive game form Γ that is not equivalent to some free perfect information extensive game form is in (i). Any Γ that is equivalent to some free perfect information extensive game form Γ' , is in (ii) when, in Γ' , at most one player has multiple choices, and is in (iii) when, in Γ' , at least two players have multiple choices.

5 Conclusion

We defined a stability index for local effectivity functions and showed its connection to the Nakamura Number of simple games. We proved that for any unstable maximal local effectivity function the index is either 2 or 3. Since strategic game forms induce maximal local effectivity functions, our result may be viewed as a rather negative one. This study constitutes a first step in the comprehension of the nature of instability. It would be interesting to compute the stability index of neutral and anonymous effectivity functions: this is a challenging combinatorial exercise. The same definition of the stability index may be extended to more general interaction forms, especially to those derived from strategic game forms. The study of

their stability index would give an insight into the nature of the instability involved in the underlying equilibrium concept.

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